Pendulum with Oscillating Support

Eric Peña

November 20, 2024

Abstract

Problem A pendulum consists of a mass m and a massless stick of length l shown in Figure 1. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$. What is the general solution for the angle of the pendulum as a function of time?

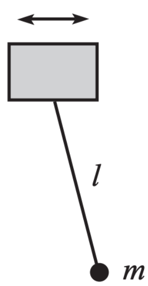

Figure 1: Pendulum with Oscillating Support

Solution

The key is to describe the position of the mass in (x, y) coordinates. This is given by:

$$(X, Y) = (x + l\sin\theta, -l\cos\theta)$$

or with time dependent variables:

$$(X(t), Y(t)) = (x(t) + l\sin(\theta(t)), -l\cos(\theta(t)))$$

We can take a derivative of the position vectors to obtain the velocity (or the square of velocity):

$$V^2 = \dot{X}^2 + \dot{Y}^2 = l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta$$

The Lagrangian can then be written as:

$$\mathcal{L} = \frac{1}{2}m(l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\sin\theta$$

The Euler-Lagrange equation for θ is therefore:

$$\frac{d}{dt}(ml^2\dot{\theta} + ml\dot{x}\cos\theta) = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$$

The equation of motion is given by:

$$l\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta$$

When we plug in the given expression for $x(t)$:

Equation of Motion of Mass m

$$l\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0$$